Contradiction within Paraxial Wave Optics and its Solution within a Particle Picture

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Abstract: It is shown that the condition provided by paraxial wave optics for the resonance frequencies of the eigenmodes of an optical resonator leads to a contradiction, if the resonator is divided into subcavities. Moreover, the results obtained in this way imply a violation of energy conservation. Paraxial wave optics seems not to allow for a solution of this problem. However a solution seems to be possible within a particle picture of the optical resonator as presented recently by the author. It is based on a consideration of the change of momentum of a photon bouncing between two mirrors with vanishing distance. This leads to a transverse force exerted by the mirrors on the photon. Assigning a relativistic mass to the photon leads to a Schrödinger equation for the transverse motion of the photon. In this way the transverse modes can be understood as transverse quantum mechanical eigenfunctions of a single photon. Additionally, the particle picture provides a physical explanation for the analogy between paraxial wave optics and the harmonic oscillator as described previously.

Keywords: Laser theory, Laser resonators, Quantum optics, Photonics, Coherence, Resonator modes

1. Introduction

As well known, the computation of the resonance frequencies ω_{qnm} of the eigenmodes of a resonator is based on the condition that the total phase shift after one round-trip must be an integer multiple of 2π. After taking into account the Gouy phase shift to compute the total phase shift, this condition delivers, according to paraxial wave optics (PWO), the relation

$$\omega_{qnm} = \frac{2\pi cq}{2L} + (1+\mathrm{n}+\mathrm{m})\frac{c}{L}\arccos\sqrt{\left(1-\frac{L}{R_1}\right)\left(1-\frac{L}{R_2}\right)}\ , \tag{1}$$

in case of an empty resonator with spherical end mirrors, see Eq. (19.23) in [1]. In Eq. (1) q is the axial mode index, L is the length of the resonator, R_1 and R_2 are the radii of curvature of the mirrors, and n and m are the transverse mode orders. The second term at the right hand side accounts for the Gouy phase shift. It is shown that Eq. (1) leads to a contradiction, if the resonator is divided into subcavities. Moreover, the obtained results seem to imply a violation of energy conservation. Since the derivation of Eq. (1) seems to be correct within PWO it is not expected that this problem can be solved within this approach.

In this paper a solution is proposed within a particle picture of the optical resonator as presented recently by the author [2], which is slightly modified in the present paper. This particle picture shows that the transverse modes can be understood as the transverse quantum mechanical eigenfunctions of a single photon. Additionally, it provides a physical explanation for the analogy between paraxial wave optics and the harmonic oscillator as described previously [3].

To derive the particle picture the change of the momentum of a photon bouncing between two mirrors with small distance is considered. In case of vanishing distance between the mirrors, this leads to a transverse force exerted on the photon. In this way it can be shown that the photon is moving within a transverse quantum mechanical potential. Assigning a relativistic mass to the photon leads to a Schrödinger equation for the transverse motion of the photon.

The possibility to describe the transverse motion of a photon in a resonator by the use of a Schrödinger equation has already been outlined by the author earlier [4] in some different context. In [4] the following Schrödinger equation for the photon has been derived

$$\left[\frac{\hbar}{2M}\Delta + \hbar\left(\omega-\omega_q\right) - V(x,y)\right]\psi(x,y) = 0\ . \tag{2}$$

Here $|\psi|^2$ is the transverse position probability density of the photon, and ω, the angular frequency of the photon. ω_q is defined by

$$\omega_q = \pi qc / L \tag{3}$$

The corresponding axial wave length is given by

$$\lambda_q = \frac{2L}{q} = \frac{2\pi c}{\omega_q}\ . \tag{4}$$

In case of an empty resonator with two end mirrors, $\hbar\omega_q$ is the associated plane wave energy of the photon in the limiting case of plane end mirrors. The quantity

$$M = \hbar\omega_q / c^2 = \hbar\frac{\pi q}{cL} = \frac{2\hbar\pi}{c\lambda_q} \tag{5}$$

is the corresponding relativistic mass of the photon.

2. Contradiction within paraxial wave optics

As is well known, all important properties, such as spot size w(z), curvature R(z) of the phase front, and Gouy phase shift ψ(z), of a Gaussian beam, are determined by the Rayleigh range z_R and the waist spot size w_0, as follows

$$w(z) = w_0\sqrt{1+\left(\frac{z}{z_R}\right)^2}\ , \tag{6a}$$

$$R(z) = z + \frac{z_R^2}{z}\ , \tag{6b}$$

$$\psi(z) = \arctan\frac{z}{z_R}\ . \tag{6c}$$

See for instance Eqs. (17.5) in [1]. In Eqs. (6) z is the distance from the waist of the beam. Therefore, all empty spherical resonators with given w_0 and z_R have identical transverse mode structures independent of the position z of the end mirrors along the resonator axis as long as the curvatures of the mirrors are given by the Eq. (6b). To look at this in more detail, we consider a resonator with one planar and one spherical end mirror as shown in

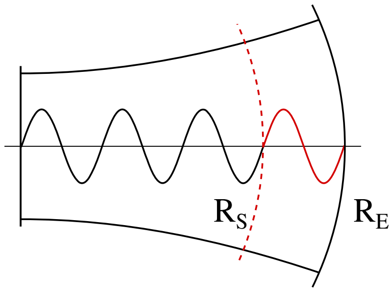


Fig. 1

Fig. 1. We now assume that one of the transverse modes propagates from the planar to the end mirror with curvature R_E. After it is reflected it propagates back, but is now reflected at a totally reflecting equiphase surface S, which is inserted at the position z_S. Since there is no loss, the wave continues to be reflected forth and back between S and the end mirror. If S is positioned at a nodal surface of the wave[1], it seems to be obvious that this wave represents a standing wave in the subcavity SC, made up by S and the end mirror. Therefore it can be expected that the local phase and intensity distribution of this wave are not changing during a round-trip through SC. Consequently, according to the generally accepted definition of an eigenmode, see for instance [1] Chapt. 14, the wave propagating in SC can be considered to be an eigenmode of SC. Basically, this is comparable with any other vibrating system. For instance, if one considers a string vibrating in an overtone, there are nodes where the string does not move. Thus, its behavior is not changed if its transverse motion is fixed at one of these nodes. Therefore, a vibration, representing an eigenmode of the full string, automatically represents eigenmodes of the subresonators created by the fixation of the string at a node. However, according to PWO, the wave propagating in the subcavity SC is not an eigenmode of SC, since it does not have a resonance frequency belonging to SC. This turns out as follows:

According to Eq. (1) the resonance frequencies of the full cavity are given by

$$\omega_{qnm} = \frac{2\pi c}{\lambda_q} + (1+\mathrm{n}+\mathrm{m})\frac{c}{L}\arccos\sqrt{1-\frac{L}{R_E}} = \frac{2\pi c}{\lambda_q} + (1+\mathrm{n}+\mathrm{m})\frac{c}{L}\arcsin\sqrt{\frac{L}{R_E}}\ . \tag{7}$$

[1] Since S is a totally reflecting surface but not a physical mirror, this condition is not really necessary. The wave makes a phase jump of π at S anyway, and a standing wave would be created nevertheless.

If we now assume that the distance between S and the end mirror is λ_q, we obtain for the resonance frequencies of the subcavity SC from Eq. (1)

$$\omega_{qnm,SC} = \frac{2\pi c}{\lambda_q} + (1+\mathrm{n}+\mathrm{m})\frac{c}{\lambda_q}\arccos\sqrt{\left(1+\frac{\lambda_q}{R_S}\right)\left(1-\frac{\lambda_q}{R_E}\right)} \ . \tag{8}$$

For $\lambda_q << R_S$, R_E and $R_S \approx R_E$ this transforms into

$$\omega_{qnm,SC} \approx \frac{2\pi c}{\lambda_q} + (1+\mathrm{n}+\mathrm{m})\frac{c}{\lambda_q}\arcsin\frac{\lambda_q}{R_E} \approx \frac{2\pi c}{\lambda_q} + (1+\mathrm{n}+\mathrm{m})\frac{c}{R_E} \ . \tag{9}$$

Thus, it turns out that the resonance frequencies of the full cavity and the subcavity are different in contradiction to the above-made assumption that the wave passes unchanged through the surface S from the full cavity into the subcavity SC, and then, according to the above argument, generates an eigenmode in the subcavity SC. Moreover, if it is assumed that Eq. (9) is correct, it follows that the resonance frequencies diminish with increasing curvature R_E, or equivalently with increasing L. Since the subcavity can be positioned at any distance L from the waist, this results in a violation of the law of energy conservation, since a photon cannot change its energy when it propagates away from the waist in free space. Since the derivation of Eq. (19.23) in [1] seems to be correct within PWO, it cannot be seen, how this problem can be solved within this approach. In the following a solution is proposed within a particle picture of the optical resonator.

3. A particle picture of the optical resonator

3.1. Derivation of a Schrödinger equation for a photon moving between equiphase surfaces

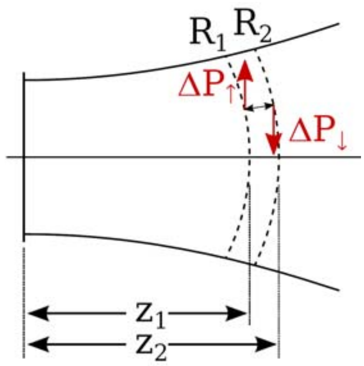


Fig. 2

To develop a particle picture of the optical resonator we consider a cavity made up by two spherical mirrors positioned close to each other at z_1 and z_2, as shown in Fig. 2. The curvatures of the two mirrors are equal to the phase front curvatures of a Gaussian beam which according to Eq. (6b) are given by

$$R_i(z_i) = z_i + \frac{z_R^2}{z_i}, \quad i \in |1,2| . \tag{10}$$

A photon bouncing back and forth between the two mirrors experiences momentum changes at both mirrors. Since the Poynting vector is oriented perpendicular to the mirrors surfaces the photon should also impinge at right angle on the mirrors surfaces. Therefore, in both cases the absolute value of the change of momentum is given by

$$\left|\Delta\vec{P}\right| = 2Mc \, . \tag{11}$$

To derive the particle picture we need the component of the momentum change perpendicular to the optical axis. For the mirror at z_1 this component is given by

$$\Delta P_\uparrow(r_1, z_1) = \frac{2Mc}{R_1} r_1 , \tag{12a}$$

and for the mirror at z_2 by

$$\Delta P_\downarrow(\mathrm{r}_2, z_2) = -\frac{2Mc}{R_2} r_2 \, . \tag{12b}$$

Here r_1 is the distance from the optical axis of the point where the photon impinges on the mirror at z_1, r_2 is the corresponding distance where the photon impinges on the mirror at z_2. The overall momentum change perpendicular to the optical axis, which a photon experiences during one round trip, is therefore

$$\Delta P_{\uparrow}(r_1,z_1)+\Delta P_{\downarrow}(r_2,z_2)=2Mc\left(\frac{r_1}{R_1}-\frac{r_2}{R_2}\right)=\frac{2Mc}{R_1R_2}(\mathrm{r}_1\,R_2-r_2\,\mathrm{R}_1)=\frac{2Mc}{R_1R_2}\left[r_2\left(R_2-R_1\right)-R_2\left(r_2-r_1\right)\right] \tag{13a}$$

which delivers

$$\lim_{R_1\to R_2}\Delta P_{\uparrow}(r_1,z_1)+\Delta P_{\downarrow}(r_2,z_2)=\frac{2Mc}{R_2^2}\left(r_2\Delta R-R_2\Delta r\right) \tag{13b}$$

Since photons impinging on a mirror exert a pressure on the mirror, the mirror must exert a force on the photons to compensate this pressure. Though it is easy to derive a relation between the momentum change of a continuously acting quantity like an electromagnetic wave and a force, this is not straightforward in the case of a single photon. To establish a relation between the change of a momentum and a force, it is necessary to define a time span during which the momentum change takes place. Though it is not possible to establish a fixation in time for the change of the momentum of a single photon, it can be stated that a photon undergoes one change of momentum at the mirror at z_1 and a second change of momentum at the mirror at z_2 during the time t_{round} which a photon takes for a round-trip. The latter is given by

$$t_{round}=\frac{2}{c}(z_2-z_1)=\frac{2}{c}\Delta z\,. \tag{14}$$

Thus, it follows from Eqs. (13b) and (14) that for $R_1\approx R_2\approx R$ the time averaged force exerted on the photon during one round trip is given by

$$K(\mathrm{r},z)=\frac{Mc^2}{R^2}\left(r\frac{\Delta R}{\Delta z}-R\frac{\Delta r}{\Delta z}\right). \tag{15}$$

In the limit $\Delta z\to 0$ the differential quotient dR/dz is obtained from Eq. (6b) as

$$\frac{dR}{dz}(z)=1-\frac{z_R^2}{z^2}\,. \tag{16}$$

For dr/dz a simple geometrical consideration delivers

$$\frac{dr}{dz}(z)=\frac{r}{R(z)} \tag{17}$$

Since for $\Delta z\to 0$ the problem of a fixation in time of the change of the momentum becomes irrelevant, the force given by Eq. (15) can be considered as a real force in this limit. After inserting Eqs. (16) and (17) into Eq. (15) this force is obtained as

$$\lim_{\Delta z\to 0}K(\mathrm{r},\mathrm{z})=\frac{Mc^2r}{R^2(z)}\left(1-\frac{z_R^2}{z^2}-\frac{R(z)}{r}\frac{r}{R(z)}\right)=-\frac{Mc^2r}{R^2(z)}\frac{z_R^2}{z^2} \tag{18}$$

which is valid for all values of z. Replacing here R(z), according to Eq. (6b), delivers

$$K(\mathrm{r},z)=-M\left(\frac{cz_R}{z^2+z_R^2}\right)^2 r\,. \tag{19}$$

Thus, we obtain that the photon is moving within a parabolic potential well given by

$$V(r,z)=\frac{1}{2}M\omega_t^2(z)r^2 \tag{20}$$

with

$$\omega_t(z)=\frac{cz_R}{z^2+z_R^2}\,. \tag{21}$$

Inserting Eq. (20) into Eq.(2) delivers

$$\left[\frac{\hbar}{2M}\Delta_t + E - \frac{1}{2}M\omega_t^2(z)(x^2+y^2)\right]\psi(x,y,z) = 0 \tag{22}$$

with

$$E(z) = \hbar\left[\omega(z) - \omega_q\right]. \tag{23}$$

Eq. (22) is identical with the Schrödinger equation of the two-dimensional harmonic oscillator. The only difference between Eq. (22) and the Schrödinger equation for a particle with rest mass m > 0 is that the potential given by Eqs. (20) and (21) only depends on the parameters λ_q and z_R, and not on the rest mass of a particle, which vanishes in the case of a photon. The solutions of Eq. (22) are well known.

With respect to Eq. (21) it shall be pointed out that ω_t does not depend on the mass of the photon. It only depends on the curvature of the phase front and its z dependence. The latter, however, is independent of the wavelength λ_q, since z_R is independent of λ_q, according to Eq. (19.4) in [1]. Consequently, this also holds for ω_t. Thus, it can be expected that ω_t can also be derived by a geometric optics approach. This will be shown in Sect. 4.

It may be argued that the above given derivation of a Schrödinger equation for the transverse motion of the photon is only valid if the phase fronts are represented by real mirrors, since otherwise no real force is exerted on the photon. Concerning this objection two arguments can be indicated. First, the properties of a propagating wave are not changed, if it is assumed that this wave is bouncing between phase fronts with vanishing distance. Therefore, the motion of a photon propagating with this wave also cannot be changed if the photon is bouncing in the same way. However, if the photon is bouncing in this manner a transverse force must be exerted on the photon, otherwise it would move away to infinity in transverse direction. Second, a real transverse force must obviously be exerted on the propagating wave by the end mirrors of the cavity, since the wave represents a propagating mass, which otherwise would move away to infinity in the same way as the photon. But this force cannot be exerted on the end mirrors only, since otherwise no smooth propagating wave would be generated. The assumption of a transverse force acting continuously between the mirrors solves this problem. The requirement of a transverse force acting on the mass of a propagating wave has not been considered so far, since it cannot be taken into account within wave optics.

Thus, it can be expected that the properties of the propagating wave can be predicted in agreement with PWO, if the motion of the photon is being considered. This is shown below except for a problem with the axial phase shift.

Finally, since the derivation of a potential describing the transverse motion of the photon, as given above, is not restricted to spherical phase fronts, but can be applied to phase fronts of any shape, it can be concluded that a transverse quantum mechanical motion of the photon described by a Schrödinger equation can be generally attributed to any photon moving with a wave having distinct phase fronts. Therefore, instead of the spherical surfaces introduced above also parabolic surfaces could have been used, which represent the mathematically exact description of the equiphase surfaces of Gaussian beams. But it is interesting that the spherical surfaces deliver results in agreement with PWO as shown below.

3.2. Computation of the transverse eigenfunctions

The eigenfunctions of Eq. (22) are given by

$$\psi_{n,,m}(x,y,z) = \sqrt{\frac{2}{\pi}}\frac{1}{w_p(z)\sqrt{2^{n+m}n!m!}}H_n\left(\frac{\sqrt{2}x}{w_p(z)}\right)H_m\left(\frac{\sqrt{2}y}{w_p(z)}\right)\exp\left(-\frac{x^2+y^2}{w_p^2(z)}\right). \tag{24}$$

Here w_p^2, where p refers to particle, is given by

$$w_p^2(z) = \frac{2\hbar}{M\omega_t(z)}. \tag{25}$$

According to Eq. (24), w_p represents the value of r, where the position probability density of a photon in the ground level drops to exp(-2). After insertion of Eq. (21) we obtain from Eq. (25)

$$w_p^2(z) = \frac{2\hbar z_R}{Mc}\left[1+\left(\frac{z}{z_R}\right)^2\right]. \tag{26}$$

which delivers

$$w_p^2(0) = \frac{2\hbar}{Mc} z_R = \frac{\lambda_q}{\pi} z_R . \tag{27}$$

This is in agreement with Eq. (17.4) in [1] obtained by the use of PWO. Insertion of Eq. (27) into Eq. (26) delivers

$$w_p(z) = w_p(0)\sqrt{1+\left(\frac{z}{z_R}\right)^2} \tag{28}$$

which is in agreement with Eq. (6a).

This result shows that, within the presented particle picture, the photons' position probability density in the whole resonator, computed by the use of Eqs. (24), (27) and (28), is for all transverse modes in agreement with the normalized intensity distribution provided by PWO.

3.4. Computation of the eigenvalues

The eigenvalues of Eq. (22) are given by

$$E_{nm}(z) = \hbar\omega_t(z)\left(n+m+1\right). \tag{29}$$

After inserting Eq. (21), this delivers

$$\omega_{qnm}(z) = \frac{2\pi c}{\lambda_q} + (n+m+1)\frac{c z_R}{z^2 + z_R^2} . \tag{30}$$

This is in agreement with Eq. (9) which can be seen after insertion of Eq. (6b) into Eq. (9) and replacing L by z. But again the problem arises that the law of energy conservation is violated, since the angular frequency of the propagating photon depends on z. However, within the particle picture, a solution seems to be possible as described in the following.

3.5. The energy balance of a photon in a propagating beam

To solve the problem with energy conservation arising from Eq. (30), the following solution is proposed. If a propagating beam wave is focused by a mirror or a lens, the energy of a photon propagating with the beam remains unchanged, due to energy conservation. But part of its energy is converted into the energy of a quantum mechanical transverse motion of the photon, dependent on its distance from the beam waist and on the spot size at the beam waist. Due to this energy conversion, the wave-optics property of a propagating photon is changed, and its wavelength increases or decreases, when it approaches the beam waist or propagates away from it. However, the total energy of the photon propagating in this way remains unchanged. Depending on the amount of energy that is converted into the quantum mechanical transverse motion, the photon can be found in different eigenstates described by Eq. (24). However, the total energy of the photon does not depend on the mode order.

Since in case of a photon the mass represented by the energy of the transverse motion is not small compared with the total mass, as in the case of a particle with non-vanishing rest mass, the question arises, whether the total mass still can be used in the Schrödinger equation describing the transverse motion of the photon. It is assumed that this question can be answered positively, since the result, obtained in this way, is confirmed by a geometric optics approach which does not involve the mass of the photon. The latter is shown in Sect. 4.

In this context it seems to be useful to introduce, for the modified wavelength of the photon, the term local wavelength, $\lambda_l(z)$, which shall be defined as twice the z dependent distance of two points where the electrical field vector goes to zero. For $\lambda_l(z)$ we obtain, according to the proposed solution of the energy problem, from the Eq. (30) the expression

$$\lambda_l(z) = \left[\frac{1}{\lambda_q} - \frac{(n+m+1)\,z_R}{2\pi\left[z^2 + z_R^2\right]}\right]^{-1} . \tag{31}$$

This delivers $\lambda_l = \lambda_q$ for $z \to \infty$. Therefore, for $z \to \infty$, energy and wavelength of the photon are equal to the values initially defined in Eqs. (3) and (4), describing a photon in the associated plane wave in the limiting case of plane end mirrors. For z = 0 we obtain

$$\lambda_l(\mathrm{z}=0) = \left[\frac{1}{\lambda_q} - \frac{(n+m+1)}{2\pi z_R}\right]^{-1} = \frac{\lambda_q}{1-\Lambda} . \tag{32}$$

with

$$\Lambda = \frac{\lambda_q}{2\pi z_R}(n+m+1) \tag{33}$$

According to these equations, $\lambda_l(0)$ is greater than λ_q for $\Lambda<1$. For $\Lambda=1$, $\lambda_l(0)$ goes to infinity, which means that the whole energy of the photon is converted into the energy of transverse motion. For $\Lambda>1$, λ_l would become negative. But this case is excluded by the requirement that the energy of the transverse motion of the photon cannot be greater than its total energy. However, even for n=m=0 the expression 1-Λ can deliver a negative value for $\lambda_l(0)$ in case of $z_R < \lambda_q/2\pi$. Therefore $z_R = \lambda_q/2\pi$ seems to represent a limiting value for z_R. The latter is confirmed by the following consideration.

For $z_R = \lambda_q/2\pi$ we obtain from Eq. (21) for the transverse oscillation frequency of the photon

$$\omega_t(z=0) = \frac{2\pi c}{\lambda_q} \,. \tag{34}$$

which, according to Eq. (4), is equal to the frequency ω_q of the corresponding axial wave. Therefore, even for a photon in the transverse ground level, the total energy of this photon completely transforms into the energy of the transverse oscillation for $z_R = \lambda_q/2\pi$. This confirms that $z_R = \lambda_q/2\pi$ represents a limit for z_R. For this limit, according to Eq. (27), the corresponding waist spot size is obtained as

$$w_p(z=0) = \frac{\lambda_q}{\sqrt{2\pi}} \tag{35}$$

This delivers for the standard deviation of the transverse position probability of the photon

$$\sigma_x(z=0) = \frac{\lambda_q}{2\sqrt{2\pi}} \,. \tag{36}$$

according to Eq. (24). Therefore according to the uncertainty principle, which claims $\sigma_x\sigma_P \geq \hbar/2$, we obtain for the standard deviation of the momentum of the photon

$$\sigma_P(z=0) \geq \frac{\sqrt{2}\pi\hbar}{\lambda_q} \,. \tag{37}$$

If we now take into account that according to Eq. (5) the momentum of the photon is given by $P=2\hbar\pi/\lambda_q$, it follows that the uncertainty of the transverse momentum is of the same magnitude as the total momentum. Consequently, a photon with a well defined energy only can propagate with a gaussian wave with $z_R >> \lambda_q$, and not with a nearly spherical wave as described by $z_R < \lambda_q/2\pi$, though at first glance, this seems to be in contradiction with spontaneous emission from a small atom.

If we divide Eq. (31) by λ_q, substitute m=0, and introduce the dimensionless quantities

$$s = \frac{z}{z_R}, \tag{38}$$

$$z_C = \frac{\lambda_q}{z_R}, \tag{39}$$

we obtain

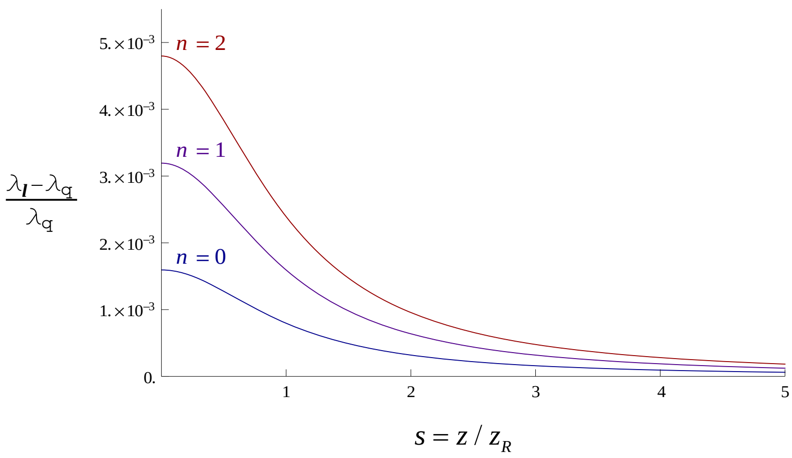


Fig. 3

$$\frac{\lambda_l}{\lambda_q} = \left[1 - \frac{(n+1)}{2\pi(s^2+1)} \frac{\lambda_q}{z_R}\right]^{-1} \approx 1 + \frac{(n+1)}{2\pi(s^2+1)} z_C \,. \tag{40}$$

In most cases the step described by " $\approx$ " is a very good approximation, since usually z_C<<1 holds. Eq. (40) shows that, under this condition, $(\lambda_l - \lambda_q)/\lambda_q$ increases proportionally to z_C. Fig. 3 shows $(\lambda_l - \lambda_q)/\lambda_q$ as a function of s for m=0, and z_C= 0.01.

The z dependence of λ_l induces a phase shift between z = 0 and z = ∞ . However, this phase shift depends on z_R in contradiction to Eq. (6c), which claims a phase shift of $\pi/2$ between z = 0 and z = ∞ independently of z_R. Concerning this contradiction it may be argued that Eq. (6c) is a mathematical consequence of the paraxial wave equation, and therefore must be correct. But, since according to Eq. (27), z_R is proportional to the beam waist spot size w_0, an intuitive consideration of this issue leads to the conclusion that the phase shift cannot be independent of w_0 . To confirm this we consider that the fundamental mode approaches a plane wave for $w_0 \to \infty$. According to Eq. (6c) the fundamental mode has a phase shift of $\pi/2$ between z = 0 and z = ∞ as long as w_0 has a finite value, but for $w_0 \to \infty$ this phase shift suddenly jumps to zero. This can not be in agreement with the physical reality. The deeper reason for this contradiction seems to be that the Helmholtz equation assumes a constant propagation vector k independent of z, and therefore does not take into account that the energy of the photon is partially converted into a transverse quantum mechanical motion. Finally, this also leads to the contradiction described in Sect. 1. Therefore, in case a focused wave, it seems to be necessary to assume that the propagation vector depends on the distance from the beam waist.

The result $\lambda_l \to \lambda_q$ for $z \to \infty$ does not necessarily indicate that all transverse modes in a real cavity have the same output frequency. Since in a real cavity the mirrors usually represent nodal surfaces of the mode, and since the position of these nodal surfaces depends on the phase shift, λ_q may assume slightly different values to adjust these nodal surfaces to the mirrors surfaces.

The proposed solution of the problem of energy conservation seems to be in contradiction to the mode beating effect of the transverse modes, which has been experimentally observed, [1] Sect. 19.3. However this is not the case. Since the quantum mechanics eigenfunctions have different local wavelengths λ_l within $-z_R \prec z \prec z_R$, dependent on the mode order, a beating coupling of their phases occurs, if the position probability density of the photon is described by a superposition of more than one eigenfunction. This leads to a fluctuating amplification in the gain medium, and finally results in the beating power output, which is observed.

Based on the above proposed explanation for the z dependence of the local wavelength in a resonator, the experiments carried through by L.G. Gouy in 1890 [5], now have a theoretical physics explanation as follows. Due to the focusing mirror in Gouys experiment part of the photons' energy is converted into a quantum mechanical transverse motion, which causes an increased local wavelength in the focus. This finally leads to the "bulls-eye" interference pattern [1] observed in the focus, when the focused beam overlaps with the unfocused beam. Thus, Gouys' experiment, and the above given theoretical explanation, seem to show that the wavelength of a photon is not a fixed quantity, but locally depends on the focusing of the beam. Moreover, Gouys' experiment could be used to prove, whether the phase shift between z = 0 and z = ∞ is independent of the beam waist spot size according to Eq. (6b), or not, as discussed above.

4. Geometric optics description of the transverse motion of the photon dependent on the distance from the beam waist

As is well known, the harmonic motion of a classical particle with mass m is described by the Hamiltonian

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2} m\,\omega^2 \left(x^2 + y^2\right). \tag{41}$$

The solution of this equation describes a periodic sinusoidal motion of a particle with angular frequency ω. By the use of Schrödingers' correspondence rules Eq. (41) can be converted into the corresponding quantum mechanical Hamiltonian of the harmonic oscillator.

Since a similar correspondence principle holds for the relationship between geometric and wave optics, it should be possible to establish a description of the transverse motion of the photon by the use of a ray tracing approach. Based on the correspondence principle, this would establish a confirmation of the results obtained above within a quantum mechanics particle picture.

For this purpose we again consider the cavity proposed in Fig. 2, with mirrors at the positions z_1 and z_2, and with radii equal to the radii of the local phase front, and assume that a ray is bouncing back and forth between these mirrors. According to Eq. (15.48) in [1], the series of points where such a ray consecutively hits the mirror at z_2 is given by

$$x_n = x_0 \cos n\theta + s_0 \sin n\theta \ . \tag{42}$$

For the sake of simplicity, the consideration has been restricted to one dimension. In Eq. (42) x_0 is the initial ray position and s_0 the initial ray slope. The two terms in Eq. (42) can be summarized into

$$x_n = A\cos(n\theta + \gamma) \tag{43}$$

with

$$A = \sqrt{x_0^2 + s_0^2}\ , \tag{44}$$

and

$$\tan\gamma = \frac{x_0}{s_0}\ . \tag{45}$$

The time a ray takes for a round-trip is again given by Eq. (14). We can therefore express the reflection points over a time axis as follows

$$x_n = A\cos\left(\frac{\theta}{t_{round}} n t_{round} + \gamma\right). \tag{46}$$

as shown by Fig. (15.15) in [1]. According to Eq. (15.45) in [1], θ is given by

$$\theta = \arccos\left(S = \frac{A+D}{2}\right) = \arcsin\sqrt{1-S^2}\ . \tag{47}$$

which delivers

$$\lim_{\Delta z \to 0} \theta = \sqrt{1-S^2}\ . \tag{48}$$

In Eq. (47) A and D are elements of the round-trip matrix of the propagating ray. The latter is given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & \Delta z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -2/R_1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \Delta z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2/(R_1 + R'\Delta z) & 1 \end{bmatrix} \tag{49}$$

with R'=dR/dz, according to Eq. (16). From Eqs. (48) and (49) we obtain, after a small amount of algebra

$$\lim_{t_{round}\to 0} \frac{\theta}{t_{round}} = \lim_{\Delta z\to 0} \frac{c\theta}{2\Delta z} = \lim_{\Delta z \to 0} c\sqrt{\frac{1-S^2}{4\Delta z^2}} = c\sqrt{\frac{1-R'}{R^2}} = \frac{c z_R}{zR} = \frac{c z_R}{z^2 + z_R^2}\ . \tag{50}$$

Insertion of this result into Eq. (46) delivers

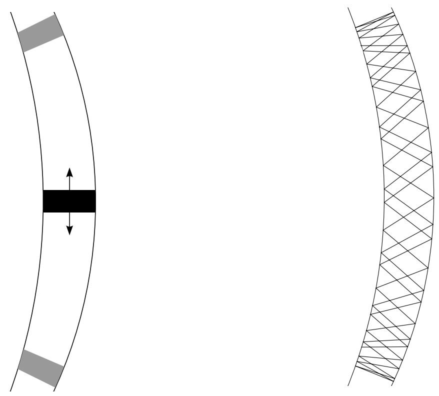
$$\lim_{n t_{round}\to t} x = A\cos\left(\omega_{ray}(z) t + \gamma\right). \tag{51}$$

with

$$\omega_{ray}(z) = \frac{c z_R}{z^2 + z_R^2}\ . \tag{52}$$

Thus, we obtain the important result that $\omega_t(z)$, given by Eq (21), is identical with the geometric optics quantity $\omega_{ray}(z)$ as already indicated in Sect. 3.1. To state this in words, the geometric optics photon, described by a bouncing ray, moves with the same transverse oscillation frequency as obtained for the quantum mechanical photon based on a consideration of the change of momentum at the mirrors. Since a mass M, according to Eq. (5), can be also attributed to the geometric optics photon, this additionally confirms that a force according to Eq. (19) is exerted on this photon.

In this way the assumption of a transverse quantum mechanical motion of the photon is confirmed by a geometric optics approach. A schematic illustration of both approaches is shown in Fig. 4. The left picture shows the photon as a particle squeezed between two mirrors with vanishing distance. The right picture shows a zig-zag photon bouncing back and forth between two mirrors.



quantum mechanics approach *ray tracing approach*

Fig. 4

5. Summary and conclusions

It has been shown that the condition provided by paraxial wave optics (PWO) for the resonance frequencies of the eigenmodes of a resonator leads to a contradiction, if the resonator is divided into subcavities. Moreover, the results obtained in this way imply a violation of energy conservation. Since the derivation of the resonance condition provided by PWO seems to be correct within this approach a solution of this problem is not expected within PWO.

In this paper a solution is proposed within a particle picture of the optical resonator, as presented recently by the author [2], which has been slightly modified in the present paper. This particle picture shows that the transverse modes can be understood as the transverse quantum mechanical eigenfunctions of a single photon. Additionally, it provides a physical explanation for the analogy between paraxial wave optics and the harmonic oscillator as described previously [3].

To derive the particle picture the change of the momentum of a photon bouncing between two mirrors with small distance has been considered. In the case of vanishing distance between the mirrors, this leads to a transverse force exerted on the photon. In this way it can be shown that the photon is moving within a transverse quantum mechanical potential. Assigning a relativistic mass to the photon leads to a Schrödinger equation for this transverse motion of the photon.

However, it turned out that both physical models, the PWO as well as the particle picture, violate of the law of energy conservation, as shown in Sections 2. and 3.4. In the PWO the problem arises from the Eq. (9), and in the particle picture from Eq. (30), which give the frequencies of the transverse modes. Both equations show that these frequencies depend on the distance z from the beam waist and decrease with increasing z. This leads to a decreasing energy of a propagating photon in a transverse mode with increasing z, and, therefore to a violation of the law of energy conservation. In the particle picture a solution of this problem has been proposed as follows. The total energy of the photon is constant in the whole cavity, but due to the focusing of the beam, part of the energy of the photon is converted into a quantum mechanical transverse motion of the photon. Due to this energy conversion, the wave-optics property of a propagating photon is changed, and its wavelength increases or decreases, when it approaches the beam waist or propagates away from it. Depending on the amount of energy converted into the quantum mechanical transverse motion, the photon can be found in different eigenstates described by Eq. (24).

Based on the above described results the Gouy effect [5] finds a theoretical physics explanation in the following way. Due to the focusing mirror in Gouys experiment, part of the photons' energy is converted into a quantum mechanical transverse motion, which causes an increased local wavelength in the focus, and finally leads to the "bulls-eye" interference pattern observed in the focus, when the focused beam overlaps with the unfocused beam. Thus, Gouys' experiment and the given theoretical explanation show that the wavelength of a photon is not a fixed quantity, but locally depends on the focusing of the beam. Since the Gouy effect also has been observed in case of focused phonon-polaritons [6], it may be possible to extend the particle approach to resonant phonons, by attributing an energy quantum to the propagating phonon.

A further important result is that only a single photon is involved to establish the particle picture. This leads to the conclusion that the reflection at the mirrors enables a single spontaneously emitted photon to build up a resonant state in a cavity with macroscopic dimensions. From this result it can be further concluded that this single resonant photon generates a second resonant photon by stimulated emission, and so forth, which finally leads to a coherent state. Therefore only a single photon in a resonant state seems to be necessary to generate a laser mode. Hence the particle picture seems to explain why it is possible that a laser mode can develop in a cavity by spontaneous emission combined with induced emission.

However, the particle picture provides a further important result as well. Since it shows that the intensity distribution in the cavity can be interpreted as the position probability density of a particle represented by the photon, it turns out that the laser resonator represents a macroscopic quantum mechanical system. In this way the particle picture shows a new facet of the particle wave dualism introduced by Schrödinger into physics, which was first proven for the photon by Einsteins' photoelectric effect.

Through further research, it shall be investigated how the particle picture can be extended to resonators with aspheric mirrors, and/or with internal elements. In case of mirrors with limited transverse extension, it should be possible to show that the effect of diffraction corresponds in the particle picture to the quantum mechanical tunnel effect. It may be further expected that the particle picture will generally be important to model photonic systems, since analogies between micro optics and quantum mechanics have already been reported by other authors [7]. For instance, it is expected that the transverse quantum mechanical motion of a photon in a step index fiber can be modeled by the use of a rectangular potential well of finite depth.

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